Nonexistence of positive supersolutions to a class of quasilinear elliptic system

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***Abstract*—**This paper discusses a class of quasilinear elliptic system in some exterior domain in R^N. Using some test functions and the method in proving an improved Hardy inequality, we give a sufficient condition under which positive supersolutions to this class of quasilinear elliptic system do not exist in some exterior domain in R^N.

***Keywords**-supersolution; Hardy potential; elliptic system.*

I. INTRODUCTION

In this paper, we discuss nonexistence of supersolution of quasilinear elliptic system

$$\begin{cases} -\Delta_p u - \mu_1 \dfrac{u^{p-1}}{|x|^p} = \dfrac{v^{q_1}}{|x|^{\sigma_1}}, \\ -\Delta_q v - \mu_2 \dfrac{v^{q-1}}{|x|^q} = \dfrac{v^{p_1}}{|x|^{\sigma_2}}, \end{cases} \quad (1)$$

in some exterior domain B_ρ^c in R^N ,where $1 < p, q < N,\ q_1 > q-1,\ p_1 > p-1,\ \mu_1$, μ_2 , σ_1 and σ_2 are any real numbers, $\rho > 1$ and $B_\rho^c = \{\mathrm{x} \| \mathrm{x}| < \rho\}$.

Definition 1. we say (u,v) is a positive supersolution(lowersolution) to (1) in B_ρ^c if (u,v) satisfies the following conditions:

$$0 < u \in W_{loc}^{1,p}(B_\rho^c) \bigcap C(B_\rho^c),$$

$$0 < v \in W_{loc}^{1,q}(B_\rho^c) \bigcap C(B_\rho^c)$$

and for any

$$0 < \varphi_1 \in W_c^{1,p}(B_\rho^c) \bigcap C(B_\rho^c),$$

$$0 < \varphi_2 \in W_c^{1,q}(B_\rho^c) \bigcap C(B_\rho^c),$$

$$\begin{cases} \int_G (|\nabla u|^{p-2} \nabla \varphi_1 \nabla u - \mu_1 \dfrac{u^{p-1}\varphi_1}{|x|^p} - \dfrac{v^{q_1}\varphi_1}{|x|^{\sigma_1}})dx \ge (\le) 0, \\ \int_G (|\nabla v|^{q-2} \nabla \varphi_2 \nabla v - \mu_2 \dfrac{v^{q-1}\varphi_2}{|x|^q} - \dfrac{u^{p_1}\varphi_2}{|x|^{\sigma_2}})dx \ge (\le) 0, \end{cases} \quad (2)$$

where $W_c^{1,p}(B_\rho^c) = \{\mathrm{w} \in W_{loc}^{1,p}(B_\rho^c), \sup p(w) \subset B_\rho^c\}$.

Nonexistence of positive supersolution to elliptic system without Hardy potential can be seen in [1,2]. The author in [3] discuss nonexistence of positive supersolution to (1) with $p = q = 2$ and Hardy potential.

Nonexistence of positive supersolution to the following equation

$$-\Delta_p w - \mu \frac{w^{p-1}}{|x|^p} = C \frac{w^q}{|x|^\sigma} \quad (3)$$

in some exterior domain B_ρ^c was studied in [4]. The author in [4] proved the solutions to (3) satisfies a comparison principle, then gave some sufficient conditions for nonexistence of positive supersolution to (3) from Picone identity in [5], Hardy inequality in [6] and the supersolution(lowersolution) to the equation

$$-\Delta_p w - \mu \frac{w^{p-1}}{|x|^p} = 0. \quad (4)$$

Definition 2.([4]) we say w is a supersolution(lowersolution) to (4) in B_ρ^c if w satisfies the following conditions: $0 < w \in W_{loc}^{1,p}(B_\rho^c) \bigcap C(B_\rho^c)$,and for any $0 < \psi \in W_c^{1,p}(B_\rho^c) \bigcap C(B_\rho^c)$,

$$\int_{B_\rho^c} (|\nabla w|^{p-2} \nabla \psi \nabla w - \mu \frac{w^{p-1}\psi}{|x|^p})dx \ge (\le) 0.$$

Let w be the positive lowersolution to (4) in B_ρ^c. If there exists a sequence $\{\theta_n\} \in W_c^{1,p}(R^N)$ such that $\lim_{n\to\infty}\theta_n = 1$ and $\lim_{n\to\infty}\int_{B_\rho^c} R(\theta_n w, w)dx = 0$, then w is called a small lowersolution to (4), where

$$R(\theta_n w, w) = |\nabla(\theta_n w)|^p - \nabla\frac{(\theta_n w)^p}{w^{p-1}}|\nabla w|^{p-2}\nabla w.$$

Motivated by [1] and [4], we prove the positive supersolutions to (1) in B_ρ^c do not exist.

The coupling of u and v makes Hardy inequality invalid in proving nonexistence of positive supersolution to (1). Borrowing the test function in [4], we overcome the above difficulty and reach our aim.

Denote by α_- the smallest solution to

$$-\gamma|\gamma|^{p-2}(\gamma(p-1)+N-p) = \mu_1$$

with $1 < p < N$, $\mu_1 \le (\frac{N-p}{p})^p$ and by β_- the smallest solution to

$$-\gamma|\gamma|^{q-2}(\gamma(q-1)+N-p) = \mu_2$$

with $1 < q < N$, $\mu_2 \le (\frac{N-q}{q})^q$ ([4]).

Theorem 1. **Let** $1 < p, q < N$, $q_1 > q-1$ and $p_1 > p-1$. If one of the following two conditions holds:

(1). $\mu_1 > (\frac{N-p}{p})^p$ or $\mu_2 > (\frac{N-q}{q})^q$

(2). $\mu_1 \le (\frac{N-p}{p})^p$, $\mu_2 \le (\frac{N-q}{q})^q$,

$$N + \frac{\beta_-}{2}(q_1+1) + \frac{\alpha_-}{2}(p_1+1) - \frac{1}{2}(\sigma_1+\sigma_2) > 0,$$

then the positive supersolution to (1) do not exist.

II. SOME LEMMAS

First, we list two results in [4].

Lemma 1. **Let** $1 < p < N$, $\mu_1 \le (\frac{N-p}{p})^p$. Then $u = r^\gamma$ is a small lowersolution to

$$-\Delta_p u - \mu_1\frac{u^{p-1}}{|x|^p} = 0 \qquad (5)$$

when $\mu_1 < (\frac{N-p}{p})^p$, $\gamma \le \alpha_- < \alpha_* = \frac{p-N}{p}$ and $u = r^\gamma$ is a small lowersolution to (5) when $\mu_1 = (\frac{N-p}{p})^p$, $\gamma \le \alpha_- = \alpha_*$.

Lemma 2([4]). Let $1 < q < N$, $\mu_2 \le (\frac{N-q}{q})^q$. Then $v = r^\gamma$ is a small lowersolution to

$$-\Delta_q u - \mu_2\frac{v^{q-1}}{|x|^q} = 0 \qquad (6)$$

when $\mu_2 < (\frac{N-q}{q})^q$, $\gamma \le \beta_- < \beta_* = \frac{q-N}{q}$ and $v = r^\gamma$ is a small lowersolution to (6) when $\mu_2 = (\frac{N-q}{q})^q$, $\gamma \le \beta_- = \beta_*$.

Next, we give a result of the supersolution to (1).

Lemma 3. Assume that (ϕ_p, ϕ_q) is a positive supersolution(lowersolution) to (1). Then for any

$$0 < \varphi_1 \in W_c^{1,p}(B_\rho^c)\cap C(B_\rho^c),$$

$$0 < \varphi_2 \in W_c^{1,2}(B_\rho^c)\cap C(B_\rho^c),$$

$$\varepsilon_{\mu_1}(\phi_1) \ge (\le)\int_{B_\rho^c} R_p(\phi_1,\phi_p)dx + \int_{B_\rho^c}\frac{v^{q_q}\phi_1^p}{\phi_p^{p-1}|x|^{\sigma_1}}dx,$$

$$\varepsilon_{\mu_2}(\phi_2) \ge (\le)\int_{B_\rho^c} R_q(\phi_2,\phi_q)dx + \int_{B_\rho^c}\frac{u^{p_1}\phi_2^q}{\phi_q^{q-1}|x|^{\sigma_2}}dx,$$

where

$$\varepsilon_{\mu_1}(\phi_1) = \int_{B_\rho^c}|\nabla\phi_1|^p dx - \mu_1\int_{B_\rho^c}\frac{\phi_1^p}{|x|^p}dx,$$

$$\varepsilon_{\mu_2}(\phi_2) = \int_{B_\rho^c}|\nabla\phi_2|^q dx - \mu_2\int_{B_\rho^c}\frac{\phi_2^q}{|x|^q}dx,$$

$$R_p(\phi_1,\phi_p) = |\nabla\phi_1|^p - \nabla(\frac{\phi_1^p}{\phi_p^{p-1}})|\nabla\phi_p|^{p-2}\nabla\phi_p,$$

$$R_q(\phi_2,\phi_q)=|\nabla\phi_2|^p-\nabla(\frac{\phi_2^q}{\phi_q^{q-1}})|\nabla\phi_q|^{q-2}\nabla\phi_q.$$

Proof. Similarly to the proof of Proposition A.2 in [4], we get Lemma 2.3 from Picone indentify.

Based on Lemma 2.1- 2.2 and Theorem 2.1 in [4], we obtain the following lemma.

Lemma 4. Assume that (u,v) is a positive supersolution to (1) . Then there exist two positive constants C_1 and C_2 such that

$$u\geq C_1|x|^{\alpha_-}\ (x\in B_{2\rho}^c) \qquad (7)$$

if $\mu_1\leq(\frac{N-p}{p})^p$ and

$$v\geq C_2|x|^{\beta_-}\ (x\in B_{2\rho}^c) \qquad (8)$$

if $\mu_2\leq(\frac{N-q}{q})^q$.

III. Proof of Theorem 1

Case 1 $\mu_1>(\frac{N-p}{p})^p$ or $\mu_2>(\frac{N-q}{q})^q$

Without loss of generality, we assume that $\mu_1>(\frac{N-p}{p})^p$. Let (u,v) is a positive supersolution to (1). Then u is a positive supersolution to

$$-\Delta_p u-\mu_1\frac{u^{p-1}}{|x|^p}=0. \qquad (9)$$

From Proposition 2.2 in [4], we know that the positive supersolutions to (3.1) do not exist. Therefore 1 has not positive supersolutions.

Case 2 $\mu_1\leq(\frac{N-p}{p})^p$, $\mu_2\leq(\frac{N-q}{q})^q$ and

$$N+\frac{\beta_-}{2}(q_1+1)+\frac{\alpha_-}{2}(p_1+1)-\frac{1}{2}(\sigma_1+\sigma_2)>0.$$

Suppose that (u,v) is a positive supersolution to (1). Then it follows from Lemma 3 that for any

$$0<\varphi_1\in W_c^{1,p}(B_\rho^c)\cap C(B_\rho^c),$$

$$0<\varphi_2\in W_c^{1,q}(B_\rho^c)\cap C(B_\rho^c),$$

$$\int_{B_\rho^c}|\nabla\phi_1|^p dx-\mu_1\int_{B_\rho^c}\frac{\phi_1^p}{|x|^p}dx\geq\int_{B_\rho^c}\frac{v^{q_1}\phi_1^p}{u^{p-1}|x|^{\sigma_1}}dx, \qquad \textbf{(10)}$$

$$\int_{B_\rho^c}|\nabla\phi_2|^q dx-\mu_2\int_{B_\rho^c}\frac{\phi_2^q}{|x|^q}dx\geq\int_{B_\rho^c}\frac{u^{p_1}\phi_2^q}{v^{q-1}|x|^{\sigma_2}}dx. \qquad \textbf{(11)}$$

Summing (10) and (11), then using Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\int_{B_\rho^c}|\nabla\phi_1|^p dx-\mu_1\int_{B_\rho^c}\frac{\phi_1^p}{|x|^p}dx\\&+\int_{B_\rho^c}|\nabla\phi_2|^q dx-\mu_2\int_{B_\rho^c}\frac{\phi_2^q}{|x|^q}dx\\&\geq 2\int_{B_\rho^c}v^{\frac{1}{2}(q_1-q+1)}u^{\frac{1}{2}(p_1-p+1)}\phi_1^{\frac{p}{2}}\phi_2^{\frac{q}{2}}|x|^{-\frac{1}{2}(\sigma_1+\sigma_2)}dx\\&\geq 2C_2^{\frac{1}{2}(q_1-q+1)}C_2^{\frac{1}{2}(p_1-p+1)}\\&\times\int_{B_{2\rho}^c}r^{\frac{\beta_-}{2}(q_1-q+1)+\frac{\alpha_-}{2}(p_1-p+1)-\frac{1}{2}(\sigma_1+\sigma_2)}\phi_1^{\frac{p}{2}}\phi_2^{\frac{q}{2}}dx\end{aligned} \qquad (12)$$

for any

$$0<\varphi_1\in W_c^{1,p}(B_\rho^c)\cap C(B_\rho^c),$$

$$0<\varphi_2\in W_c^{1,q}(B_\rho^c)\cap C(B_\rho^c).$$

In the sequence, we prove

$$\begin{aligned}&\int_{B_\rho^c}|\nabla\phi_1|^p dx-\mu_1\int_{B_\rho^c}\frac{\phi_1^p}{|x|^p}dx\\&+\int_{B_\rho^c}|\nabla\phi_2|^q dx-\mu_2\int_{B_\rho^c}\frac{\phi_2^q}{|x|^q}dx\\&<2\int_{B_{2\rho}^c}r^{\frac{\beta_-}{2}(q_1-q+1)+\frac{\alpha_-}{2}(p_1-p+1)-\frac{1}{2}(\sigma_1+\sigma_2)}\phi_1^{\frac{p}{2}}\phi_2^{\frac{q}{2}}dx\end{aligned}$$

By choosing ϕ_1 and ϕ_2 in (12). Set $r=|x|$ and let $R\geq 2\rho$. Select a cutoff function in [4]

$$\theta_R=\begin{cases}0, & r<\frac{3\rho}{2},\\ \frac{2r}{\rho}-3, & \frac{3\rho}{2}\le r<2\rho,\\ 1, & 2\rho\le r\le R,\\ \frac{\ln\frac{R^2}{r}}{\ln R}, & \mathrm{R}\le r\le R^2,\\ 0, & R^2\le r.\end{cases}$$

Choose κ_1 and κ_2 such that $\kappa_1=1$ when $p\ge 2$, $\kappa_1>\frac{2}{p}$ when $1<p<2$, $\kappa_2=1$ when $q\ge 2$, $\kappa_2>\frac{2}{q}$ when $1<q<2$.

From Lemma 1-2, we know that r^{α_-} is a small lowersolution to (5) and r^{α_-} is a small lowersolution to (6). The proof of Proposition C.1 and Proposition A.2 in [4] implies that for any $\phi_1=r^{\alpha_-}(\theta_R)^{\kappa_1}$, $\phi_2=r^{\beta_-}(\theta_R)^{\kappa_2}$,

$$\int_{B_\rho^c}|\nabla\phi_1|^p dx-\mu_1\int_{B_\rho^c}\frac{\phi_1^p}{|x|^p}dx \le C_3+\int_{R<|x|\le R^2}R_p(\phi_1,r^{\alpha_-})dx \le C_3+C_4\frac{R^{p\alpha_-+N-p}}{(\ln R)^p}, \quad \textbf{(13)}$$

$$\int_{B_\rho^c}|\nabla\phi_2|^q dx-\mu_2\int_{B_\rho^c}\frac{\phi_2^q}{|x|^q}dx \le C_5+\int_{R<|x|\le R^2}R_q(\phi_2,r^{\beta_-})dx \le C_5+C_6\frac{R^{q\beta_-+N-q}}{(\ln R)^q}, \quad \textbf{(14),}$$

where C_3, C_4, C_5, C_6 are some positive constants independent of R.

(13)-(14), Lemma 1 and Lemma 2 lead to

$$\int_{B_\rho^c}|\nabla\phi_1|^p dx-\mu_1\int_{B_\rho^c}\frac{\phi_1^p}{|x|^p}dx+\int_{B_\rho^c}|\nabla\phi_2|^q dx-\mu_2\int_{B_\rho^c}\frac{\phi_2^q}{|x|^q}dx \le C_3+C_4\frac{R^{p\alpha_-+N-p}}{(\ln R)^p}+C_5+C_6\frac{R^{q\beta_-+N-q}}{(\ln R)^q} \to C_3+C_5 (R\to\infty). \quad \textbf{(15)}$$

On the other hand, from the definitions of θ_R, ϕ_1 and ϕ_2, we have that if

$$N+\frac{\beta_-}{2}(q_1+1)+\frac{\alpha_-}{2}(p_1+1)-\frac{1}{2}(\sigma_1+\sigma_2)>0,$$

then

$$2C_2^{\frac{1}{2}(q_1-q+1)}C_1^{\frac{1}{2}(p_1-p+1)}\times\int_{B_{2\rho}^c} r^{\frac{\beta_-}{2}(q_1-q+1)+\frac{\alpha_-}{2}(p_1-p+1)-\frac{1}{2}(\sigma_1+\sigma_2)}\phi_1^{\frac{p}{2}}\phi_2^{\frac{q}{2}}dx$$
$$\ge 2C_2^{\frac{1}{2}(q_1-q+1)}C_1^{\frac{1}{2}(p_1-p+1)}R^{\frac{q\beta_-}{2}+\frac{p\alpha_-}{2}}\times\int_{2\rho\le|x|\le R} r^{\frac{\beta_-}{2}(q_1-q+1)+\frac{\alpha_-}{2}(p_1-p+1)-\frac{1}{2}(\sigma_1+\sigma_2)}dx$$
$$\ge 2C_2^{\frac{1}{2}(q_1-q+1)}C_1^{\frac{1}{2}(p_1-p+1)}R^{\frac{q\beta_-}{2}+\frac{p\alpha_-}{2}}[C_7R^{N+\frac{\beta_-}{2}(q_1-q+1)+\frac{\alpha_-}{2}(p_1-p+1)-\frac{1}{2}(\sigma_1+\sigma_2)}-C_8]$$
$$\to+\infty(R\to\infty), \quad (16)$$

where C_7 and C_8 are some positive constants independent of R.

(15) and (16) imply that (13) holds if R is large enough. (12) contradicts with (13). Thus, the supersolutions to (1) do not exist.

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