The Sphere of Influence Graph: Theory and Applications

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Abstract:

The sphere of influence graph of a set of points in the plane is a graph \( G(V, E) \) in which the vertex set \( V \) consists of the points, and the edge set \( E \) consists of edges joining two points if their nearest neighbor circles intersect. The nearest neighbor circle of a point \( P \) is the largest circle centered at \( P \) that does not contain any other points in its interior. This graph was proposed in 1980 as a geometric model for a primal sketch in computer vision. Since then it has been explored, generalized, and applied to problems in several disciplines. This paper traces the history of this graph, surveys the progress made since 1980, and lists areas for further research.

Keywords - sphere of influence graph; proximity graph; intersection graph; graph theory; computational geometry, computer vision, artificial intelligence

I. INTRODUCTION

In 1980 I proposed a planar graph that I called the sphere of influence graph (SIG) as a computational model of the primal sketch in computer vision [24]. The SIG was motivated by the fact that earlier proximity graphs such as the minimum spanning tree (MST) and the relative neighbourhood graph (RNG), that were used in this context, necessarily yield connected graphs [25]. To force these graphs to produce disconnected components where appropriate, heuristics were needed to delete the longest edges from these graphs, thus requiring parameters to be tuned. The SIG on the other hand produces disconnected components that agree to a remarkable degree with human perception completely automatically without the need of tuning parameters. The SIG of seven points is illustrated in Fig. 1, where each point \( x \) is surrounded by the largest disc with center \( x \), that contains no points in its interior.
For any point \( v_i \) in Fig. 1, the radius of its corresponding disc is determined by the nearest neighbor of \( v_i \). The SIG contains an edge between vertices \( v_i \) and \( v_j \) if, and only if, the discs associated with \( v_i \) and \( v_j \) intersect. The SIG in the example of Fig. 1 contains two disconnected components, and one cycle between vertices \( v_i, v_k, \) and \( v_j \). The example in Fig. 1 is a plane graph, but examples may be constructed in which the SIG contains complete subgraphs (cliques) that contain crossing edges. Thus the SIG need not be a plane graph. Since 1980 the SIG has been explored from the computational geometry and graph theory points of view, it has been generalized in several ways, and it has found use in several applications. This paper traces the history and progress made on this graph and its generalizations, and lists open problems for further research.

II. THEORY

A. The Size of the Sphere of Influence Graph

Although the SIG in the plane contains sub-graphs that are cliques [15], I observed that the cliques could not be very large, which led me to conjecture in 1980 that for \( n \) points in the plane the number of edges in their SIG was \( O(n) \). Shortly after, in 1982, Avis and Horton [1] proved that the SIG of \( n \) points in the plane has at most \( 29n \) edges, and that every decision tree algorithm for computing a SIG requires at least \( \Omega(n \log n) \) steps in the worst case. Hosam ElGindy, one of our graduate students at McGill University at that time, pointed out that with existing computational geometry techniques the SIG could be computed in \( O(n \log n) \) time, thus matching the lower bound [1]. Soss showed that a result obtained independently by Reifenberg [18] and Bateman and Erdős [2] could be used to show that the SIG has at most \( 18n \) edges. Michael and Quint [MQ-1994] reduced this upper bound to \( 17.5n \), and Soss sharpened this upper bound to \( 15n \) [21]. David Avis conjectured that a tight upper bound is \( 9n \) [21]. Soss also obtained bounds on the size of the open sphere of influence graph in \( L_\infty \) metric spaces [20]. Guibas, Pach, and Sharir [7] showed that for fixed dimensions the number of edges in the SIG remains \( O(n) \). Palka and Sperling [16] consider the SIG of a set of points generated by a Poisson random process on the real line. They determine the expected number and variance of connected components (clusters) formed by sets of \( n \) points. Another interesting property of random SIGs is the number of vertices of degree one. Sperling determines the expected value and variance of this number for SIGs in \( d \)-dimensional spaces for \( d \geq 2 \) [19]. A natural way to construct a random SIG is to generate the points uniformly on the unit hypersphere or in a unit hypercube. For the hypersphere Dwyer [5] found upper and lower bounds on the expected number of edges in a random SIG. Since then asymptotically exact values for the expected number of edges in random SIGs for points in the unit hypercube have been determined for all values of \( d \) [4]. See also the papers of Furedi [6] and Hitczenko, Janson, and Yukich [9].

Related to the problem of cliques in SIGs is that of determining which complete graphs are SIGs. It is known that \( K_8 \) is a SIG [28], and it was shown by Kézdy and Kubicki [27], that \( K(12) \) is not a closed SIG.

B. Generalizations of the Sphere of Influence Graph

The original definition of the SIG with the Euclidean distance [24] has been generalized to other metrics [3], [7], [13], [14]. Also, the original definition of a SIG that specified open balls, was modified slightly to include the boundaries of the balls, leading to the closed SIG [8]. In the context of computing the surfaces of dense clouds of
points, Klein and Zakhirman [33] proposed the k-SIG, which constructs the balls for each point with radius determined by the point’s kth nearest neighbor. A broader generalization is the abstract SIG, which is a graph that is isomorphic to a SIG [8]. Holm and Bogart [26] generalize the SIG by assigning to each point a tolerance, and adding an edge between two points if their balls overlap by more than the sum (also minimum) of their tolerances. Lipman has introduced the maximum tolerance SIG [29]. Jacobson, Lipman, and McMorris [30], characterize trees which are open and closed SIGs, and obtain a bound on the number of edges when the SIGs are triangle-free. Their characterization of closed SIGs is succinct: a tree is a closed SIG if, and only if, it contains a perfect matching. For more on trees see [39]. McMorris and Wang [35] proposed a variant of the SIG, which they called the sphere-of-attraction graph (SAG). In this generalization the points are divided into two sets C and P, which may be viewed, respectively, as customers and products, and the ball for each point in C has a radius equal to the closest point in P. In the SAG an edge is inserted connecting two points in C if their balls intersect.

III. APPLICATIONS

The main area of application for SIGs is in low-level computer vision [12], [17] cluster analysis [10], pattern recognition [23], geographic information systems [37], modeling visual illusions [23], and streaming processes in music perception [22]. For some of these applications it is useful to know which graphs are SIGs of sets of pixels in an image. In this context Lipman [11] shows that every SIG has a realization as a set of points with integer coordinates. One weakness of the SIG in some applications, where a connected shape structure must be extracted from a dot pattern, is evident when there are pairs of points closer to each other than to the rest of the set. In such circumstances these pairs will be connected to each other but not to the remaining points. Klein and Zachmann [33] propose several extensions of the SIG to handle such problems. One of their extensions constructs balls with radius greater than that determined by the nearest neighbor. The sphere-of-attraction graph proposed by McMorris and Wang [35] has applications to marketing [28].

IV. CONCLUSION AND OPEN PROBLEMS

A natural family of open problems concerns sharpening the bounds on the number of edges in a SIG. For example, it is not known whether the complete graphs K9, K10, or K11 are SIGs. Soss showed that the Euclidean open SIG has at most 15n edges. Can this bound be improved to 9n? Another area where the exploration of the SIG can make a contribution is in the visualization of large graphs or networks, by computing the SIG of the graph, and rendering the SIG instead of the large graph, as has been done with other proximity graphs [38].

Algorithms for computing the SIG of a prescribed graph already exist [31], [32]. Computing the SIG of social networks [34] could also provide a useful tool for analyzing “social cartography” [36] processes such as citation networks, commercial networks and marketing, customer relations, and the spreading of diseases.

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